**LEBANESE AMERICAN UNIVERSITY**
**Department of Computer Science and Mathematics**
**Calculus IV**
**Exam I Spring 2013 (March 13, 2013)**

**Name:** Solutions **ID:**
**Circle your section: Dr M. Hamdan Dr L. Issa**

| Question Number | Grade |
|---|---|
| 1. 8 % | |
| 2. 8% | |
| 3. 8% | |
| 4. 12% | |
| 5. 12% | |
| 6. 10% | |
| 7. 10% | |
| 8.16% | |
| 9.16% | |
| Total | |

1. (8%) What is the largest value that the directional derivative of $f(x,y) = xy^2$ can have at $(1,1,1)$?

$|\vec{\nabla} f| = \sqrt{1+4} = \sqrt{5}$

$f_x = y^2 = 1$

$f_y = 2xy = 2.$

$\vec{\nabla} f = \vec{i} + 2\vec{j}$

$D_u f = |\vec{\nabla} f| \cos\theta$

largest $D_{\vec{u}} f$ is $|\vec{\nabla} f|$

2. (8%) Find the equation of the plane tangent to $x^2 - y - 5z = 0$ at the point $(2,-1,1)$

$\vec{\nabla} f = 4\vec{i} - \vec{j} - 5\vec{k}$

$f_x = 2x = 4$

$f_y = -1$

$f_z = -5$

$\Rightarrow$ plane :

$4(x-2) - (y+1) - 5(z-1) = 0.$

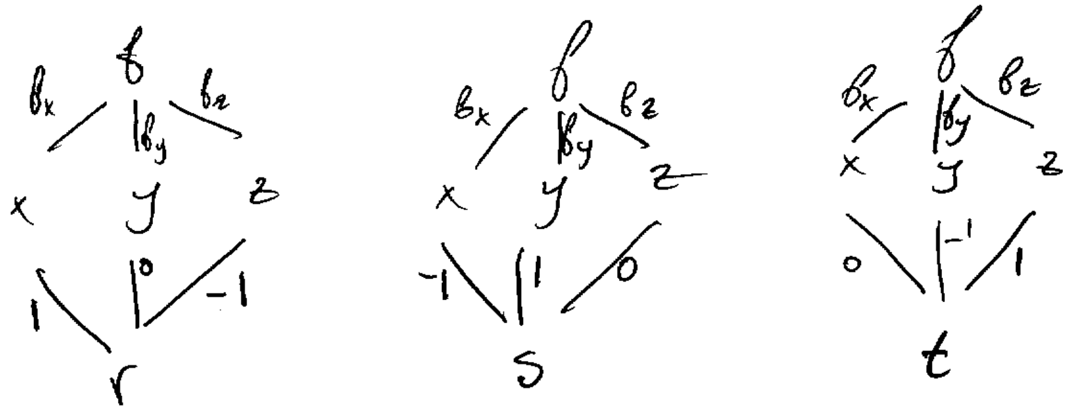
3. (8%) Use implicit differentiation to find $\frac{dy}{dx}$ at the point $(-1,1)$ for $4x - \frac{4}{y} + 6x^2y^2 = 0$

$$F_x = 4 + 12xy^2 = -8$$

$$F_y = +\frac{4}{y^2} + 12x^2y = 16.$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{8}{16} = \frac{1}{2}$$

4. (12%) If $f(x, y, z)$ is differentiable, $x = r - s, y = s - t$, and $z = t - r$, show that $f_r + f_t + f_s = 0$



$$f_r = f_x - f_z$$

$$+ \quad f_s = -f_x + f_y$$

$$+ \quad f_t = -f_y + f_z$$

$$f_r + f_s + f_t = 0$$

5. (12%) At the point $(1,2)$, the function $f(x,y)$ has derivative equal to 2 in the direction toward the point $(2,2)$, and derivative $-2$ in the direction toward the point $(1,1)$.

(a) Find $f_x(1,2)$ and $f_y(1,2)$

$$\vec{u_1} = \vec{i}$$
$$\vec{u_2} = -\vec{j}$$

$$\boxed{\begin{array}{l} f_x = 2 \\ f_y = 2 \end{array}}$$

Since $D_{\vec{u_1}} f = f_x = 2 \quad u_1 = \vec{i}$

$$D_{\vec{u_2}} f = -f_y = -2 \Rightarrow$$

$$\boxed{\vec{\nabla} f = 2i + 2\vec{j}}$$

(b) Find the derivative of $f$ at $(1,2)$ in the direction toward the point $(4,6)$

$$\vec{u_3} = \frac{\vec{A}}{|\vec{A}|} = \frac{3\vec{i} + 4\vec{j}}{\sqrt{9+16}} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

$$D_{\vec{u_3}} f = \vec{\nabla} f \cdot \vec{u_3}$$
$$= (2i + 2j) \cdot \left(\frac{3}{5} i + \frac{4}{5}\vec{j}\right)$$
$$= \frac{6}{5} + \frac{8}{5} = \frac{14}{5}.$$

6. (10%) Approximate the value: $\sqrt{(4.9)+4} + 2.02$ using linearization,. Hint: start with the function: $f(x, y) = \sqrt{x+4} + y$ and take it form there.

Use $(x_0, y_0) = (5, 2)$.

$\Delta x = 4.9 - 5 = -0.1$

$\Delta y = 2.02 - 2 = 0.02$

$f(x,y) = \sqrt{4.9+4} + 2.02 \approx f(5,2) + f_x(5,2) * \Delta x + f_y(5,2) * \Delta y.$

$f(5,2) = \sqrt{5+4} + 2 = 5$

$f_x = \left.\frac{1}{2\sqrt{x+4}}\right|_{(5,2)} = \frac{1}{6}$

$f_y = 1$

$\therefore \quad f(x,y) \approx 5 + \frac{1}{6}(-0.1) + 1(0.02).$

7. (10%) You plan to calculate the volume inside a cylindrical pipe that is about 1 m in diameter and 2 km in length. With which measures should you be more careful? (that is, to which dimension is the sensitivity to change higher)? explain

$V = \pi r^2 h$

$V_r = 2\pi r h = 4000\pi$

$V_h = \pi r^2 = \pi$

$dV = V_r\, dr + V_h\, dh$

$= \underline{4000\pi\, dr} + \pi\, dh$

$\Rightarrow$ We should be more careful calculating the RADIUS.

8. (16%) A closed rectangular box of dimensions $x, y$,and $z$ should have volume 100cm$^3$. The cost of the material used in the box is 10 $cents/cm^2$(that is surface area) for the top and the bottom, and 12 $cents/cm^2$ for the front and back sides, and 15 $cents/cm^2$ for the front and back sides.

(a) Write the expression of the function $f(x, y, z)$ for the total cost in terms of the dimensions $x, y$ and $z$

$V = xyz = 100$

a $f(x,y,z) = 10 * 2\, xy + 12 * 2\, yz + 15 * 2\, xz$

$= 20\, xy + 24\, yz + 30\, xz$

(b) Deduce the dimensions $x, y,$and $z$ that minimize the total cost of material. (Use Lagrange. Just set up the system. Don't evaluate the final answer.)

Minimize $f(x, y, z)$ subject to the constraint $xyz = 100$

$$\vec{\nabla} f = \lambda \vec{\nabla} g.$$

$$\vec{\nabla} f = (20y + 30z)\vec{i} + (20x + 24z)\vec{j} + (24y + 30x)\vec{k}$$

$$\vec{\nabla} g = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$\therefore$ We need.

$$\begin{cases} 20y + 30z = \lambda yz \\ 20x + 24z = \lambda xz. \\ 24y + 30x = \lambda xy. \\ xyz = 100 \end{cases}$$

9. (16%) Find the absolute extrema of $f(x, y) = z = x^2 - y^2 - 2x + 4y$ over the triangular plate bounded by the x-axis, the line $y = x + 2$, and on the right by $x = 2$?

I) int. points ??

$f_x = 2x - 2 = 0$ $\qquad x = 1$

$f_y = -2y + 4 = 0$ $\qquad y = 2$

$\boxed{\text{CRITICAL Point} = (1, 2)}$

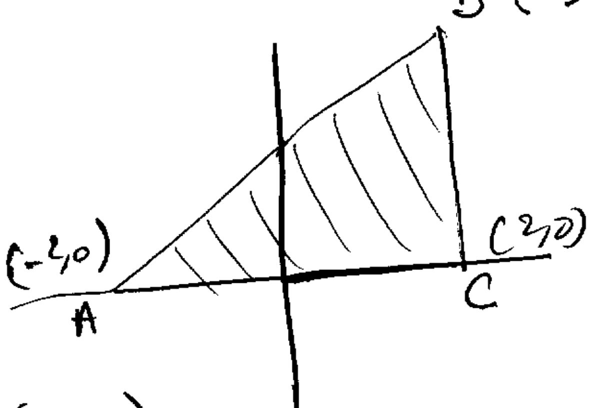
$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = -2$

$\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$

$\Rightarrow (1, 2)$ : Saddle point.




II On the Boundary ?

1) on AB: $\underline{y = x + 2}$

$z = x^2 - (x+2)^2 - 2x + 4(x+2)$

$z = \cancel{x^2} - \cancel{x^2} - 4 - \cancel{4x} - 2x + \cancel{4x} + 8 =$

$z' = -2$ $\Rightarrow$ No crit. pts there.

2) on BC: $\underline{x = 2}$ $\qquad z = -y^2 + 4y$

$z' = -2y + 4 = 0 \Rightarrow y = 2$

$\boxed{(2, 2)}$

3) on AC: $y = 0$

$z = x^2 - 2x$

$z' = 2x - 2$ $\qquad x = 1$ $\qquad \boxed{(1, 0)}$

| | $z$ |
|---|---|
| (-2,0) | 8 |
| (2,2) | 4 |
| (1,0) | -1 |
| (2,0) | 0 |

$\Rightarrow$ Abs. max = 8 @ (-2,0)

Abs min = -1 @ (-1,0)